Instructions: Legibly complete each of the following on lined paper and submit on Gradescope.

1. Let $a, b, c \in \mathbb{Z}$ be arbitrary. Prove the following.
   (a) If $a \mid b$ and $b \mid c$, then $a \mid c$.
   (b) If $a \mid b$ and $a \mid c$, then $a|(bs + ct)$ for all $s, t \in \mathbb{Z}$.

2. We proved the following in class:

   Proposition 1 (Quotient-Remainder Theorem for $\mathbb{N}_0$)**.** *Let $n, d \in \mathbb{N}_0$ with $d \neq 0$. There exist unique $q, r \in \mathbb{N}_0$ such that $n = dq + r$ and $0 \leq r < d$.*

   Finish the proof of the Quotient-Remainder Theorem for $\mathbb{Z}$ using the proposition above.[1] In other words, prove that for all $n, d \in \mathbb{Z}$ with $d \neq 0$ there exist unique $q, r \in \mathbb{Z}$ such that $n = dq + r$ and $0 \leq r < |d|$.

3. Make Cayley tables for the operations indicated below.
   (a) addition and multiplication modulo 7
   (b) addition and multiplication modulo 8

4. Let $m, n \in \mathbb{Z}$ with $m \neq 0$. Prove that if $n = mq + r$ for some $q, r \in \mathbb{Z}$, then $\gcd(m, n) = \gcd(m, r)$.

5. Use Euclid's Extended Algorithm to write each of the following as linear combinations of their arguments.
   (a) $\gcd(87, 2500)$
   (b) $\gcd(85, 105)$

6. Solve $7x - 1 \equiv 5 - 6x \pmod{70}$ for some $0 \leq x < 70$.

7. This question concerns the following proposition on the Fibonacci numbers $(F_n)_{n\geq 0}$ where $F_0 = 0$.

   Proposition 2. *Let $(A_n)_{n\geq 0}$ be a sequence of real numbers satisfying $A_n = A_{n-1} + A_{n-2}$ for all $n \geq 2$. Then $A_{n+1} = A_1 F_{n+1} + A_0 F_n$ for all $n \geq 0$.*

   (a) Use strong mathematical induction to prove the proposition.
   (b) Prove $F_{a+b+1} = F_{a+1}F_{b+1} + F_a F_b$ for all $a, b \in \mathbb{Z}_+$. **Hint**: let $A_n = F_{a+n}$ for all $n \geq 0$.
   (c) Prove $F_{2n+1} = F_{n+1}^2 + F_n^2$ for all $n \geq 0$.
   (d) Prove $F_n \mid F_{kn}$ for all $n, k \geq 1$. **Hint**: $(k + 1)n = kn + (n - 1) + 1$.

8. Study for the exam on Friday, 9th April 2021!

[1] This problem is optional, but it's a good idea to do it anyway.